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On the numerical solution of the generalized Prandtl equation using variation-diminishing splines ⋆

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Abstract

Convergence results are proved for Cauchy principal value integrals of the Schoenberg variation-diminishing splines and its first derivative. The use of such splines in the numerical solution of the Prandtl and generalized Prandtl integral equations is proposed. A Nyström-type method and a modified Nyström method are used and compared computationally.

Keywords: Approximating splines; B-splines; Cauchy principal value; Integral equations

1. Introduction

In a recent work [7], a class of approximating splines was studied in the context of numerical integration. These splines were of the form

$$\widehat{S}_n(f,x) := \sum_{i=1}^{n} f(z_{in}) B_{in}^p(x), \tag{1}$$

where the B_{in}^p are the normalized B-splines of order p which form a basis for the spline space S_{p,π_n} determined by the integer p and the vector of knots $\pi_n := \{t_{in},\ i = 1, 2, \ldots, n+p\}$, and the z_{in} are arbitrary points subject only to the condition that each z_{in} lies in the support of the corresponding B_{in}^p. A particular case of (1) occurs when the z_{in} are the so-called Schoenberg points in which case the spline is called the Schoenberg variation-diminishing (SVD) spline. This spline, which we denote by

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$S_n(f;x)$, has the interesting property that the derivative S_n' is also an approximating spline of form (1):

$$S_n'(f;x) = \sum_{i=1}^{n} f'(\eta_{in}) B_{in}^{p-1}(x). \tag{2}$$

This implies that any convergence theorem valid for $\widehat{S}_n$ will also be valid for S_n'. In particular, the uniform convergence results for Cauchy principal value (CPV) integrals of the form

$$I(\omega_{\alpha\beta} f; x) := \fint_{-1}^{1} \omega_{\alpha\beta}(t) \frac{f(t)}{t-x} \,\mathrm{d}t, \quad x \in \mathring{J} := (-1,1), \tag{3}$$

where

$$\omega_{\alpha\beta}(t) := (1-t)^{\alpha}(1+t)^{\beta}, \quad \alpha, \beta > -1, \tag{4}$$

will hold both for S_n and S_n'. This simultaneous convergence will enable us to use the SVD splines for the numerical solution of the generalized Prandtl equation (GPE)

$$\frac{\omega_{\alpha\beta}(x) f(x)}{H(x)} - \frac{1}{2\pi} I\big((\omega_{\alpha\beta} f)'; x\big) = g(x), \quad x \in \mathring{J}, \tag{5}$$

where H does not vanish in $J := [-1,1]$. While the theory will be valid for all $\alpha, \beta > 0$, in practice we shall only deal with the case when $\alpha = \beta = \frac{1}{2}$, namely the Prandtl equation (PE)

$$\frac{(1-x^2)^{1/2} f(x)}{H(x)} - \frac{1}{2\pi} I\big(((1-t^2)^{1/2} f)'; x\big) = g(x), \quad x \in \mathring{J}. \tag{5'}$$

This is not the first attempt to use splines to solve the Prandtl equation. In [6], interpolatory cubic splines with equally-spaced simple knots were used. However, it is felt that our approach gives much more flexibility.

In Section 2, we shall define the spline spaces S_{p,π_n} and prove some convergence theorems for the SVD spline and its derivative. In Section 3, we study the numerical solution of the GPE (5), and, in particular, the PE (5′), using SVD splines. To this end we shall need to prove some uniform convergence results for CPV integrals. In Section 4, we shall present computational results, both for CPV integrals and for the numerical solution of the PE. Finally, in Section 5 we shall make some concluding remarks.

2. Variation-diminishing splines

To define a spline space S_{p,π_n}, we start with a partition

$$X_m: \quad x_{0m} := -1 < x_{1m} < \cdots < x_{mm} < x_{m+1,m} := 1,$$

and a vector of positive integers $\{d_j: j = 0, \ldots, m+1\}$ where $d_0 = d_{m+1} = p$ and $d_j \leqslant p-2$, $j = 1, \ldots, m$. This restriction on d_j is necessary to insure that $S_n \in C^1(J)$ and implies that $p > 2$. We now set

$$n := \sum_{j=0}^{m+1} d_j,$$

and define $\Pi_n := \{t_{in}: i = 1, \ldots, n + p\}$ to be the nondecreasing set of elements of X_m where each element x_{jm} is repeated exactly d_j times in Π_n, $j = 0, \ldots, m + 1$. The t_{in} are the knots of the spline space S_{p,π_n} and the endpoints are p-fold knots.

A sequence of spline spaces S_{p,π_n} is said to be locally uniform if

$$\frac{x_{j+1,m} - x_{jm}}{x_{k+1,m} - x_{km}} \leqslant A, \quad k = j \pm 1,$$

where $A \geqslant 1$ does not depend on j or m. This property does not depend on the spline spaces S_{p,π_n} but only on the underlying partitions X_m, and clearly there are many spline spaces associated with the same partition. Similarly, the norm of S_{p,π_n}, H_m, which is defined by $H_m := \max_{0 \leqslant j \leqslant m}(x_{j+1,m} - x_{jm})$ depends only on X_m. In our discussion, we shall always assume that $H_m \Rightarrow 0$ as m, or equivalently n, $\Rightarrow \infty$.

For each spline space S_{p,π_n}, there exists a basis consisting of the normalized B-splines B^p_{in}, $i = 1, \ldots, n$, which have the following properties of interest to us [7]:

(1) $B^p_{in}(x) > 0$, $x \in \mathring{J}_{i,i+p} := (t_{in}, t_{i+p,n})$, $i = 1, \ldots, n$;

(2) $B^p_{in}(x) = 0$, $x \notin \mathring{J}_{i,i+p}$ except that $B^p_{1n}(-1) = B^p_{nn}(1) = 1$;

$$(3) \quad \left(\sum_{i=1}^{n} \alpha_i B^p_{in}(x)\right)' = (p-1) \sum_{i=1}^{n} \frac{\alpha_i - \alpha_{i-1}}{t_{i+p-1,n} - t_{in}} B^{p-1}_{in}(x). \tag{6}$$

For any spline $\widehat{S}_n \in S_{p,\pi_n}$ of the form (1), we have the following two results from [7].

Theorem 1. *Let $f \in C(J)$ and let $\{S_{p,\pi_n}\}$ be a sequence of spline spaces such that $H_m \Rightarrow 0$ as $n \Rightarrow \infty$. Then,*

(1) *for any sequence of splines $\widehat{S}_n$ defined by* (1) *with $z_{in} \in \mathring{J}_{i,i+p}$,*

$$\|f - \widehat{S}_n\|_\infty \Rightarrow 0, \quad \text{as } n \Rightarrow \infty; \tag{7}$$

(2) *if the sequence of spline spaces is locally uniform, then the modulus of continuity of $\widehat{S}_n$ satisfies*

$$w(\widehat{S}_n; t) \leqslant B w(f; t), \tag{8}$$

where B is a constant independent of n.

We note on this occasion that the proof of (8) in [7] is incomplete. A hopefully complete proof can be found in [9].

We now consider a special choice of the points z_{in} in (1), namely the so-called Schoenberg points

$$\xi_{in} := \frac{t_{i+1,n} + \cdots + t_{i+p-1,n}}{p-1}, \quad i = 1, \ldots, n. \tag{9}$$

The SVD spline $S_n(f; x)$ of order p is defined by

$$S_n(f; x) := \sum_{i=1}^{n} f(\xi_{in}) B^p_{in}(x). \tag{10}$$

The SVD spline has the interesting property that S_n' is an approximating spline of the form (1)

$$S_n'(f;x) = \sum_{i=1}^{n} f'(\eta_{in}) B_{in}^{p-1}(x) = \widehat{S}_n(f';x), \tag{11}$$

with $\eta_{in} \in \mathring{J}_{i,i+p-1}$, so that Theorem 1 also applies to S_n'. To see this, we substitute (10) into (6) and find that

$$S_n'(f;x) = (p-1) \sum_{i=1}^{n} \frac{f(\xi_{in}) - f(\xi_{i-1,n})}{t_{i+p-1,n} - t_{in}} B_{in}^{p-1}(x). \tag{12}$$

But since $f(\xi_{in}) - f(\xi_{i-1,n}) = (\xi_{in} - \xi_{i-1,n}) f'(\eta_{in})$, $\eta_{in} \in (\xi_{i-1,n}, \xi_{in})$ and $\xi_{in} - \xi_{i-1,n} = (t_{i+p-1,n} - t_{i,n})/(p-1)$, it follows

$$S_n'(f;x) = \sum_{i=1}^{n} f'(\eta_{in}) B_{in}^{p-1}(x) = \widehat{S}_n(f';x), \quad \eta_{in} \in \mathring{J}_{i,i+p-1}, \tag{13}$$

i.e., η_{in} lies in the support of B_{in}^{p-1}.

We thus have from Theorem 1 that if $f \in C^1(J)$, then

$$\|f - S_n\|_\infty \Rightarrow 0, \quad \|f' - S_n'\|_\infty \Rightarrow 0, \qquad \text{as } n \Rightarrow \infty, \tag{14}$$

simultaneously and if in addition the spline spaces are locally uniform, then

$$w(S_n;t) \leqslant Bw(f;t), \qquad w(S_n';t) \leqslant Bw(f';t), \tag{15}$$

for some constant B independent of n. We shall apply these results in the next section to derive uniform convergence results for certain CPV integrals which arise in connection with the numerical solution of the GPE (5).

3. The generalized Prandtl equation

We shall now discuss the GPE (5) although our computations will only be with the PE (5′). This is so for several reasons. First, it is this equation which arises in practice. Second, the general case is computationally much more complicated since the computation of $I((\omega_{\alpha\beta}B_{in}^p);x)$ involves the incomplete Beta function which is not easy to compute, whereas for $\alpha = \beta = \frac{1}{2}$ there exists an algorithm which is much simpler [2]. Third, if we use a Nyström-type method to solve (5), then when we collocate to solve for the approximate solution, two of the collocation points are -1 and 1 and we encounter difficulties for every pair of values of $\alpha, \beta > 0$ except $\alpha = \beta = \frac{1}{2}$. However, as we shall see, we can avoid these difficulties when we use a modified Nyström method.

If we carry out the differentiation in (5), we get that

$$I((\omega_{\alpha\beta}f)';x) = I(\omega_{\alpha\beta}f';x) + I(\omega_{\alpha\beta}'f;x), \tag{16}$$

with $\omega_{\alpha\beta}'(x) = \beta\omega_{\alpha,\beta-1}(x) - \alpha\omega_{\alpha-1,\beta}(x)$. In order to insure that the CPV integrals exist, we shall assume that $f \in C^1(J)$ and that $f' \in H_\mu(J)$, where $H_\mu(J) = \{g: w(g;t) \leqslant Ct^\mu, 0 < \mu \leqslant 1, C > 0\}$.

Our objective now is to show that

$$I((\omega_{\alpha\beta}S_n)';x) \Rightarrow I((\omega_{\alpha\beta}f)';x), \quad \text{as } n \Rightarrow \infty, \tag{17}$$

uniformly for all $x \in \mathring{J}$. To show (17), we state a more refined version of a theorem in [8].

Theorem 2. *Let $f \in H_\mu(J)$ for some $0 < \mu \leqslant 1$ and assume that we are given a sequence $\{f_n: n = 1,2,\ldots\}$ of approximations to f which satisfies the following three conditions:*
(1) $\|r_n\|_\infty \Rightarrow 0$ *as* $n \Rightarrow \infty$ *where* $r_n := f - f_n$;
(2) $r_n(-1) = 0$ *if* $\beta \leqslant 0$, $r_n(1) = 0$ *if* $\alpha \leqslant 0$;
(3) $r_n \in H_\sigma(J)$, $0 < \sigma \leqslant \mu$.
Then,

$$I(\omega_{\alpha\beta}r_n: x) \Rightarrow 0, \quad \text{as } n \Rightarrow \infty, \tag{18}$$

uniformly for all $x \in \mathring{J}$ provided that

$$\sigma + \gamma > 0, \quad \gamma := \min(\alpha, \beta). \tag{19}$$

Condition (2) in [8] required that $r_n(\pm 1) = 0$ for any α, β; however, by inspecting the proof in [8] it can be seen that the condition can be weakened to that given above. This modification in Theorem 2 will be important in showing uniform convergence of (17). Note that if $\gamma > 0$, then (18) holds for all $f \in H_\mu(J)$ while if $\sigma = 1$, then (18) holds for all $\omega_{\alpha\beta}$. This observation will also be relevant in our discussion of (5).

Returning to (17), we must show that

$$I(\omega_{\alpha\beta}r'_n; x) \Rightarrow 0, \quad \text{as } n \Rightarrow \infty, \tag{20}$$

$$I(\omega'_{\alpha\beta}r_n; x) \Rightarrow 0, \quad \text{as } n \Rightarrow \infty, \tag{21}$$

uniformly for all $x \in \mathring{J}$ where $r_n := f - S_n$. However, this follows from (14) and (15) and Theorem 2 with $\sigma = \mu$, since

$$S_n(f;-1) = f(\xi_{1n})B^p_{1n}(-1) = f(-1), \qquad S_n(f;1) = f(\xi_{nn})B^p_{nn}(1) = f(1),$$

by the definition of ξ_{in}, the properties of B^p_{in} and the fact that the endpoint knots have multiplicity p. Hence $r_n(\pm 1) = 0$. On the other hand, since $\alpha, \beta > 0$, there is no requirement on $r'_n(\pm 1)$. Furthermore, $f \in H_1(J)$. Hence (17) holds uniformly for all $x \in \mathring{J}$ which is a necessary condition that the solution of (5) with f replaced by S_n converges to the solution of (5).

If we were now to use a Nyström-type method to solve (5), we would replace f by S_n in the integral $I((\omega_{\alpha\beta}f)';x)$ and then collocate at the points ξ_{in} yielding the system of linear equations

$$\frac{\omega_{\alpha\beta}(\xi_{in})f(\xi_{ij})}{H(\xi_{in})} - \frac{1}{2\pi}I((\omega_{\alpha\beta}S_n)';\xi_{in}) = g(\xi_{ij}), \quad i = 1,\ldots,n. \tag{22}$$

However, if $\alpha, \beta \neq \frac{1}{2}$, the CPV integral in (22) does not exist for $\xi_{1n} = -1$ and $\xi_{nn} = 1$. Hence, we can use this method only for the PE (5′). We shall show below how to deal with the GPE using a modified Nyström method. Returning to the system (22) with $\alpha = \beta = \frac{1}{2}$, we solve this for the unknown functional values $f(\xi_{in})$ from which we construct an approximation to the solution of (5′). Computationally, this amounts to evaluating the CPV integrals

$$\int_{t_{in}}^{t_{i+p,n}} \frac{(1-t^2)^{1/2} B'_{in}(t)}{t-\xi_{jn}} \, dt \quad \text{and} \quad \int_{t_{in}}^{t_{i+p,n}} \frac{(1-t^2)^{-1/2} B_{in}(t)}{t-\xi_{jn}} \, dt,$$

for which algorithms similar to those in [2] can be constructed. Of course, if $\xi_{jn} \notin \mathring{J}_{i,i+p}$, then the integral is not a CPV integral but a regular integral except when $\xi_{jn} = t_{in}$ when we may have trouble. Actually, this only occurs when ξ_{jn} is an endpoint since in other cases we just extend the limits of the integral. However, the integral

$$\int_{-1}^{1} \frac{(1-t^2)^{-1/2} B_{in}(t)}{t+1} \, dt \tag{23}$$

is not a CPV integral although the integral

$$\int_{-1}^{1} \frac{(1-t^2)^{1/2} B_{in}(t)}{t+1} \, dt$$

is a convergent integral. To deal with (23) we write it as

$$\int_{-1}^{1} (1-t^2)^{-1/2} \frac{B_{in}(t) - B_{in}(-1)}{t+1} \, dt + B_{in}(-1) I(\omega_{-1/2,-1/2}; -1). \tag{24}$$

We can evaluate this easily since $I(\omega_{-1/2;-1/2}; x) \equiv 0$ and the first integral is regular at $t = -1$.

We note here that we do not use (11) or (12) for S'_n but instead use the formulation

$$S'_n(f; x) = \sum_{i=1}^{n} f(\xi_{in}) B'_{in}(x). \tag{25}$$

In the next section, we give some computational results for CPV integrals and for the numerical solution of (5′). These examples illustrate the advantages of using the SVD splines since we have a lot of flexibility in the placement of the knots and are not restricted to simple knots but can have knots up to multiplicity $p-2$. This in addition to the possibility of choosing splines of arbitrary order $p \geqslant 3$.

We now return to the GPE. The reason we cannot use a Nyström-type method to solve (5) is that in the Nyström extension of (5)

$$\frac{\omega_{\alpha\beta}(x) f(x)}{H(x)} - \frac{1}{2\pi} I\big((\omega_{\alpha\beta} S_n)'; x\big) = g(x), \quad x \in \mathring{J}, \tag{26}$$

there are n unknowns, namely the values $f(\xi_{in})$, $i = 1, \ldots, n$. Hence, we must collocate at a set of n points $Z := \{\tau_{jn}, \ j = 1, \ldots, n\}$ to get a system of n linear equations for the $f(\xi_{in})$. However, from (26) we see that the set of collocation points Z must coincide with the set $\{\xi_{in}\}$ since otherwise we introduce new unknowns $f(\tau_{jn})$. This leads to the problem that $I\big((\omega_{\alpha\beta} S_n f)'; \xi_{in}\big)$ is not defined for

$i = 1, n$. The way we overcome this difficulty is to modify the Nyström extension (26) by replacing f with $S_n f$ throughout (5) to get the modified Nyström extension

$$\frac{\omega_{\alpha\beta}(x)S_n(f;x)}{H(x)} - \frac{1}{2\pi}I\big((\omega_{\alpha\beta}S_n)';x\big) = g(x), \quad x \in \mathring{J}. \tag{27}$$

This modification essentially decouples the two sets $\{\xi_{in}\}$ and Z so that we are free to choose the collocation points in any way we see fit. Thus the problem of evaluating $I\big((\omega_{\alpha\beta}S_n f)';\pm 1\big)$ need not arise. In addition, we have the advantage that we can avoid collocation points close to the knots t_{in} since these points can cause some computational problems as indicated in [3].

We now give a heuristic justification for this modification which is applicable to a lesser degree to Cauchy singular integral equations. In the Nyström method, given an integral equation of second kind, say,

$$f(x) + \int_{-1}^{1} H(x, y, f(y))\,dy = g(x), \tag{28}$$

we replace $f(y)$ in the integral by an approximation, say $\sum_{i=1}^{n} a_{in}(y)f(x_{in})$, to yield the Nyström extension. Then, we collocate at the points $\{x_{in}\}$ and solve the resulting system of linear or nonlinear equations for the unknown values $f(x_{in})$. Theoretically, we could also replace $f(x)$ in (28) by $\sum_{i=1}^{n} a_{in}(x)f(x_{in})$ and then we could collocate at any set of n distinct points $\{z_{in}\}$. However, we do not do this since this would degrade the accuracy of the solutions because generally the error in the approximation of $f(x)$ will be more than the error in the integral induced by the approximation to $f(y)$, inasmuch as integration is a smoothing operation. In the Prandtl case, this is not the situation. First, the derivative of the approximation to f appears and, as is well known, the error in the derivative of the approximation is generally greater than the error in the approximation of the function. Furthermore, the CPV integral is not a smoothing operation. Hence if we replace f by S_n throughout (5), we will not cause any loss in accuracy while by doing this we would be able to deal with any $\alpha, \beta > 0$ to say nothing of the advantage gained by the possibility of choosing collocation points which yield a stable calculation.

4. Computational results

We now give some computational results to illustrate various aspects of the theory in Section 3. Most of these results are for the weight function

$$\omega_{1/2,1/2}(t) = (1 - t^2)^{1/2} \tag{29}$$

and the PE (5′) since the computations needed for other values of $\alpha, \beta > 0$ are very complicated. We first show some results related to the approximation of CPV integrals with nonregular integrands, using SVD splines. For such integrands, a suitable placement of the knots is needed.

In Table 1 we present the absolute errors of the approximation, for different poles x and increasing number of nodes n, of a CPV integral (3) with weight function (29), and $f(t) = (1-t^2)^{1/2}$. As $f(t)$ is regular but has singularities in the first derivative at $t = \pm 1$, a sequence of nonuniform meshes

Table 1

n	Mesh 1		Mesh 2		Mesh 3	
	$x = 0.2$	$x = 0.6$	$x = 0.2$	$x = 0.6$	$x = 0.2$	$x = 0.6$
10	$1.1 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	$5.8 \cdot 10^{-3}$	$3.1 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$	$5.7 \cdot 10^{-2}$
18	$5.3 \cdot 10^{-4}$	$3.4 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	$1.0 \cdot 10^{-2}$	$2.3 \cdot 10^{-3}$	$2.3 \cdot 10^{-2}$
34	$1.8 \cdot 10^{-4}$	$1.3 \cdot 10^{-3}$	$3.3 \cdot 10^{-4}$	$2.8 \cdot 10^{-3}$	$5.9 \cdot 10^{-4}$	$5.3 \cdot 10^{-3}$

performs better when more of the nodes are concentrated near the endpoints. The approximating splines used are the cubic SVD splines ($p = 4$).

In Table 1, mesh 3, mesh 2 and mesh 1, in that order, concentrate the knots more towards ± 1.

In Table 2 we present the absolute errors of the approximation, by SVD cubic splines, for different poles x of a CPV integral (3) with $\omega_{0,0}(t) \equiv 1$ and $f(t) = \exp(|t|)$. As $f(t)$ is not regular at $t = 0$, the approximation is better when $t = 0$ is a multiple node.

In Table 3 we present the absolute errors of the approximation of a CPV integral (3) with $\omega_{0,0}(t) = 1$ and $f(t) = (1 - t^2)^{1/2}$ with the aim to show the better behaviour of the formula with the increase in the order p of the SVD spline, based on the same mesh.

We now furnish an application of the above formulas in the solution of the PE (5′).

Tables 4 and 5 present absolute errors R_n in the approximation of the solution $f(t)$ of equation (5′) at the set of collocation points ξ_{in}, $i = 1, 2, \ldots, n$, in (9), using SVD cubic splines. Tables 6 and 7 show the behaviour of the solution $f(t)$ of the integral equations proposed in [6], using our algorithm with the SVD cubic splines. For these equations, we have no knowledge of the exact solution so that our only way to check the quality of the approximation is to compare it with the results in [6] and indeed our results indicate the same order of precision as in [6].

We remark that, in addition to obtaining, more or less, the same order of approximation as the methods used in [6], our algorithm is more flexible and computationally more efficient: it requires the solution of the only one linear system (22).

In Table 8, we compare some results obtained when solving (5′) using SVD splines and both the Nyström-type method and the modified Nyström method. We have denoted by $R^{(1)}$ the absolute error obtained with the Nyström-type method and with $R^{(2)}$ and $R^{(3)}$ the absolute errors with the modified method using as collocation points respectively the Schoenberg points ξ_{in} and a different set $\tau_{jn} = \frac{1}{2}(\xi_{jn} + \xi_{j+1,n})$ except when $j = [\frac{1}{2}n]$ where we used $\frac{1}{3}(\xi_{jn} + \xi_{j+1,n})$ and $\frac{2}{3}(\xi_{jn} + \xi_{j+1,n})$.

Evidently $R^{(1)}$, $R^{(2)}$ and $R^{(3)}$ are of the same order of magnitude, with a slightly better behaviour of $R^{(1)}$, as the function f in the left-hand term in (5′) is not replaced by S_n.

In Table 9 we show some numerical results obtained using the modified Nyström method for (5′) using the collocation points τ_{jn} when the solution f has singularities in the derivative. We have denoted with $R^{(1)}$ the absolute error when all the components of the vector of knots Π_n are

Table 2

x	$n = 17$	
	Simple node	Double node
0.1	$2.1 \cdot 10^{-2}$	$6.7 \cdot 10^{-3}$
0.6	$1.6 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$
0.9	$3.8 \cdot 10^{-2}$	$3.6 \cdot 10^{-2}$

Table 3

p	$x = 0.1$	
	$n = 10$	$n = 18$
2	$1.8 \cdot 10^{-2}$	$8.4 \cdot 10^{-3}$
3	$1.4 \cdot 10^{-2}$	$4.6 \cdot 10^{-3}$
4	$1.1 \cdot 10^{-2}$	$3.6 \cdot 10^{-3}$
5	$1.0 \cdot 10^{-2}$	$3.1 \cdot 10^{-3}$

Table 4

$$(1-x^2)^{1/2}f(x)-(2\pi)^{-1}\int_{-1}^{1}\frac{\left((1-t^2)^{1/2}f(t)\right)'}{t-x}\,\mathrm{d}t = x\left(1+(1-x^2)^{1/2}\right),\quad f(t)=t$$

$n=4$		$n=8$		$n=16$	
ξ	R_4	ξ	R_8	ξ	R_{16}
−1	$1.1\cdot10^{-4}$	−1	$2.6\cdot10^{-4}$	−1	$7.3\cdot10^{-5}$
−0.7778	$7.1\cdot10^{-5}$	−0.9047	$3.0\cdot10^{-5}$	−0.9555	$6.6\cdot10^{-5}$
−0.3333	$8.0\cdot10^{-5}$	−0.7142	$1.6\cdot10^{-4}$	−0.8666	$6.1\cdot10^{-5}$
		−0.4285	$4.0\cdot10^{-4}$	−0.7333	$1.8\cdot10^{-5}$
		−0.1428	$3.1\cdot10^{-4}$	−0.6000	$1.1\cdot10^{-4}$
				−0.4666	$1.0\cdot10^{-4}$
				−0.3333	$7.6\cdot10^{-5}$
				−0.2000	$2.9\cdot10^{-5}$
				−0.0666	$3.0\cdot10^{-5}$

Table 5

$$(1-x^2)^{1/2}f(x)-(2\pi)^{-1}\int_{-1}^{1}\frac{\left((1-t^2)^{1/2}f(t)\right)'}{t-x}\,\mathrm{d}t = -(1-x^2)^{3/2}-\tfrac{3}{2}(\tfrac{1}{2}-x^2),\quad f(t)=t^2-1$$

$n=4$		$n=8$		$n=16$	
ξ	R_4	ξ	R_8	ξ	R_{16}
−1	$5.7\cdot10^{-2}$	−1	$7.4\cdot10^{-3}$	−1	$2.3\cdot10^{-3}$
−0.7778	$7.6\cdot10^{-3}$	−0.9047	$2.5\cdot10^{-3}$	−0.9555	$6.5\cdot10^{-4}$
−0.3333	$5.2\cdot10^{-2}$	−0.7142	$1.0\cdot10^{-2}$	−0.8666	$2.9\cdot10^{-3}$
		−0.4285	$1.0\cdot10^{-2}$	−0.7333	$2.6\cdot10^{-3}$
		−0.1428	$9.8\cdot10^{-3}$	−0.6000	$2.3\cdot10^{-3}$
				−0.4666	$2.2\cdot10^{-3}$
				−0.3333	$2.2\cdot10^{-3}$
				−0.2000	$2.1\cdot10^{-3}$
				−0.0666	$2.0\cdot10^{-3}$

Table 6

$$(1-x^2)^{1/2}f(x)-(2\pi)^{-1}\int_{-1}^{1}\frac{\left((1-t^2)^{1/2}f(t)\right)'}{t-x}\,\mathrm{d}t = 1$$

$n=6$		$n=11$		$n=21$	
ξ	$f(\xi)$	ξ	$f(\xi)$	ξ	$f(\xi)$
		−0.8000	0.496502	−0.8000	0.501196
−0.6000	0.596919	−0.6000	0.609766	−0.6000	0.611119
		−0.4000	0.662917	−0.4000	0.663762
−0.2000	0.686631	−0.2000	0.688755	−0.2000	0.689251

distinct and with $R^{(2)}$ the absolute error when the vector Π_n has multiple (double) components at the singularities of f'. The connection between knot multiplicity and smoothness of the function f is evident.

Table 7

$$(1-x^2)^{1/2}f(x)/(2-x)-(2\pi)^{-1}\int_{-1}^{1}\frac{\left((1-t^2)^{1/2}f(t)\right)'}{t-x}\,\mathrm{d}t=|x|$$

$n=6$		$n=11$		$n=21$	
ξ	$f(\xi)$	ξ	$f(\xi)$	ξ	$f(\xi)$
		0.8000	0.369 501	0.8000	0.374 169
0.6000	0.390 829	0.6000	0.394 582	0.6000	0.396 477
		0.4000	0.357 916	0.4000	0.360 500
0.2000	0.299 511	0.2000	0.302 332	0.2000	0.306 196

5. Concluding remarks

(1) The SVD spline is a special case of the quasi-interpolatory spline treated in [4] and discussed in the context of numerical integration in [1]. It is shown in [4] that $S_n(f)$ reproduces linear functions. Result (14) follows from [4, Theorem 5.3] with $s=\ell=2$, $q=\infty$ and $r=0,1$ and result (15) can be proved as in [7,9] using [4, Theorem 5.2]. In fact, we can use the error estimates in these theorems to get the rates of convergence of S_n to f and S_n' to f'. Since $f\in C^1(J)$, we have that

$$\|f-S_n\|=\mathrm{O}(H_m\omega(f';H_m)),\qquad \|f'-S_n'\|=\mathrm{O}(\omega(f';H_m)).$$

Unfortunately, we are unable to say anything about convergence to the solution of the integral equation.

(2) Result (14) is true for all spline spaces S_{p,π_n} and not only for locally uniform spaces and, similarly, the result involving S_n in (15). Unfortunately, we have been unable to prove that the result

Table 8
$f(x)=x^2-1$

	ξ	$R^{(1)}$	$R^{(2)}$	$R^{(3)}$
$N=4$	-1	$5.7\cdot10^{-2}$	$1.3\cdot10^{-4}$	$4.2\cdot10^{-5}$
	-0.7778	$7.6\cdot10^{-3}$	$4.9\cdot10^{-2}$	$4.9\cdot10^{-2}$
	-0.3333	$5.2\cdot10^{-2}$	$1.1\cdot10^{-4}$	$1.4\cdot10^{-4}$
$N=8$	-1	$7.4\cdot10^{-3}$	$1.2\cdot10^{-4}$	$1.9\cdot10^{-4}$
	-0.9047	$2.5\cdot10^{-3}$	$9.2\cdot10^{-3}$	$9.3\cdot10^{-3}$
	-0.7142	$1.0\cdot10^{-2}$	$2.7\cdot10^{-2}$	$2.7\cdot10^{-2}$
	-0.4285	$1.0\cdot10^{-2}$	$2.7\cdot10^{-2}$	$2.7\cdot10^{-2}$
	-0.1428	$9.8\cdot10^{-3}$	$2.7\cdot10^{-2}$	$2.7\cdot10^{-2}$
$N=16$	-1	$2.3\cdot10^{-3}$	$1.4\cdot10^{-3}$	$3.6\cdot10^{-5}$
	-0.9555	$6.5\cdot10^{-4}$	$2.0\cdot10^{-3}$	$1.9\cdot10^{-3}$
	-0.8666	$2.9\cdot10^{-3}$	$6.4\cdot10^{-3}$	$5.9\cdot10^{-3}$
	-0.7333	$2.6\cdot10^{-3}$	$6.3\cdot10^{-3}$	$6.0\cdot10^{-3}$
	-0.6000	$2.3\cdot10^{-3}$	$6.2\cdot10^{-3}$	$6.1\cdot10^{-3}$
	-0.4666	$2.2\cdot10^{-3}$	$6.1\cdot10^{-3}$	$6.0\cdot10^{-3}$
	-0.3333	$2.2\cdot10^{-3}$	$5.9\cdot10^{-3}$	$5.9\cdot10^{-3}$
	-0.2000	$2.1\cdot10^{-3}$	$5.9\cdot10^{-3}$	$5.9\cdot10^{-3}$
	-0.0666	$2.0\cdot10^{-3}$	$5.9\cdot10^{-3}$	$5.9\cdot10^{-3}$

Table 9
$f(x) = \sqrt{1-x^2} - \frac{1}{2}\sqrt{3}$ for $|x| < \frac{1}{2}$, $f(x) = 0$ otherwise

	ξ	$R^{(1)}$	$R^{(2)}$
$N = 17$	-1	$2.4 \cdot 10^{-3}$	$7.9 \cdot 10^{-5}$
	-0.8750	$1.5 \cdot 10^{-3}$	$2.7 \cdot 10^{-5}$
	-0.1250	$5.7 \cdot 10^{-3}$	$2.7 \cdot 10^{-3}$
	0	$5.4 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$
$N = 33$	-1	$1.4 \cdot 10^{-3}$	$8.3 \cdot 10^{-6}$
	-0.8750	$2.5 \cdot 10^{-3}$	$2.0 \cdot 10^{-5}$
	-0.1250	$2.1 \cdot 10^{-3}$	$6.7 \cdot 10^{-4}$
	0	$2.0 \cdot 10^{-3}$	$6.5 \cdot 10^{-4}$

involving S_n' in (15) holds when S_{p,π_n} is an arbitrary spline space. Hence, we must restrict ourselves to locally uniform spline spaces when solving the Prandtl equation.

(3) Monegato and Pennacchietti [5] have studied an integration rule for $I((\omega_{\alpha\beta}f)';x)$ obtained by replacing f with an interpolating polynomial. While they are able to prove pointwise convergence, they have not proved uniform convergence. Indeed, for the PE, uniform convergence cannot be proved using Theorem 2 since in this case $\sigma = \frac{1}{2}\mu$ and since $\gamma = -\frac{1}{2}$ for $\omega_{\alpha\beta}'f$, $\sigma + \gamma$ is not greater than 0 even when $\mu = 1$ as it is in our case. Nevertheless, since $f' \in H_\mu(J)$ for some $0 < \mu \leqslant 1$ and the derivative of the interpolating polynomial is in $H_{\mu/2}$, one may still be able to prove uniform convergence since $I(\omega_{-1/2,-1/2};x) = 0$.

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